M. Abouzaid – 26/5/08 – Wrapped Fukaya categories & string topology

- (M,ω) exact symplectic $\omega = d\theta$, convex at infinity 

  ie: $\theta \rightsquigarrow$ Liouville vector field X_θ, $\omega(X_\theta, -) = \theta$, points outwards

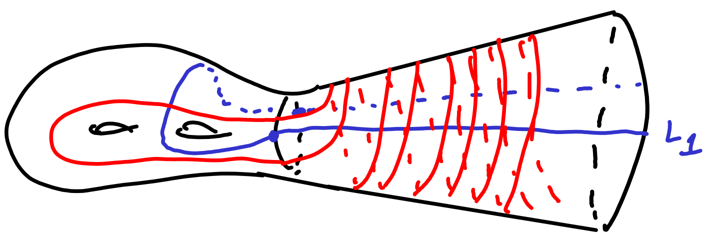
  Reeb vector field at the boundary: R_θ generates $\ker(\omega_{|\partial M})$.

- L exact Lagrangian ($\omega_{|L} = 0$, $\theta_{|L} = df$ for some $f: L \to \mathbb{R}$)

  s.t. $\partial L \subset \partial M$ Legendrian (ie. $f_{|\partial L}$ is locally constant).

$\to$ we can define a chain complex $CW^*(L_1, L_2)$ for any such L_1, L_2
(wrapped chain complex)

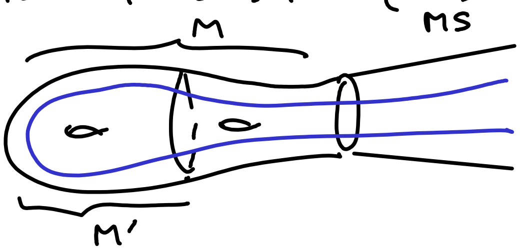
$HW^*(L_1, L_2)$ wrapped Floer homology.



ie. wrap one of L_i infinitely along Reeb flow, then take Floer complex

$\mathcal{W}(M)$ is an A_∞-cat. with objects = such L's and morphisms $CW^*(L_1, L_2)$.

<u>Q</u>: Restriction functors? ($\xleftrightarrow[MS]{}$ when $Y \subset X$, $i^*: Coh(X) \to Coh(Y)$).



<u>Thm</u> (A. – Seidel):

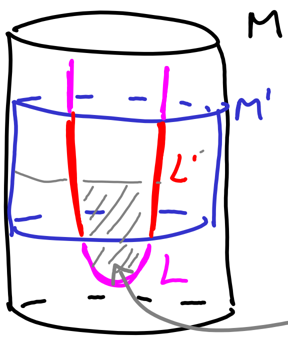
if $M' \subset M$ is a codim. 0 inclusion of exact sympl. mflds w/ convex ∂ and $L \subset M$ has the property that $f_{|L \cap \partial M'}$ is <u>constant</u> $(*)$, then $\exists$ unital A_∞-morphism $CW^*(L, L) \longrightarrow CW^*(L', L')$ where $L' = M' \cap L$.

<u>Point today:</u> we can get rid of assumption (*) if $M' \cong T^*Q$ is a tubular nbd of a compact closed exact Lagrangian submanifold $Q \subset M$.

<u>Ex:</u> M = cylinder $T^*S^1 \supset M'$ = smaller cylinder

$L' \cong \amalg$ 2 cotangent fibers

$H^*(\Omega S^1) \simeq \mathbb{C}[t,t^{-1}]$



$HW^*(L,L) = 0$

$HW^*(L',L') = Mat_{2\times 2}(\mathbb{C}[t,t^{-1}])$

$\nexists$ unital hom. $HW^*(L,L) \to HW^*(L',L')$

problem = <u>this disc</u>. ← obstruction class.

- Given Q smooth, define $\begin{cases} Ob(\mathcal{P}(Q)) = \text{points } p \in Q \\ Hom_*(p,q) = C_{-*}(\Omega_{p,q}) \end{cases}$

  chains on path space $p \rightsquigarrow q$
  composition = concatenation

Enlarge to twisted complexes:

$Tw\,\mathcal{P}(Q)$ Objects $= \left(P = \bigoplus_{i=0}^{n} p_i,\ D_P\right)$

$D_P \in Hom_*(P,P) = \bigoplus_{i,j} Hom_*(p_i, p_j)$

s.t. $\partial D_P = D_P^2$.

- We'll construct a functor $\mathcal{W}(M) \longrightarrow Tw\,\mathcal{P}(Q)$ for $Q \subset M$ exact Lagr.
  & in the case $M \simeq T^*Q$ get an equivalence of A_∞-categories $Tw\,\mathcal{W}(T^*Q) \simeq Tw\,\mathcal{P}(Q)$

- The equivalence result for $M \simeq T^*Q$ will follow from the fact that our functor maps T_p^*Q (cotangent fiber) to p, and

$CW^*(T_p^*Q, T_p^*Q) \to C_*(\Omega_{p,p})$, using

Thm (Nadler, FSS) : T_p^*Q generates $W(T^*Q)$

Thm (Abbondandolo–Schwarz) : The above map induces an isom.
$HW^*(T_p^*Q, T_p^*Q) \cong H_*(\Omega_p Q)$

- general construction:

- $L \subset M \rightsquigarrow (P_L, D_L)$ where $P_L = \bigoplus_{p_i \in L \cap Q} p_i$

  For each i,j we need $\delta_{i,j} \in C_*(\Omega_{p_i,p_j})$

  Look at $\mathcal{M}(p_i,p_j) = \left\{ \begin{array}{lll} u: D^2 \to M, & \bar\partial u = 0 & \\ -1 \bigcirc 1 & -1 \mapsto p_i & \frown \mapsto L \\ & 1 \mapsto p_j & \smile \mapsto Q \end{array} \right\} / \mathbb{R}$

  Evaluation along lower boundary gives a map $\mathcal{M}(p_i,p_j) \to \Omega_{p_i,p_j}$

  At chain level, get $\delta_{i,j} := [ev_* \mathcal{M}(p_i,p_j)] \in C_*(\Omega_{p_i,p_j})$

  $D_L = (\delta_{i,j})$ satisfies $\partial D_L = D_L^2$ because

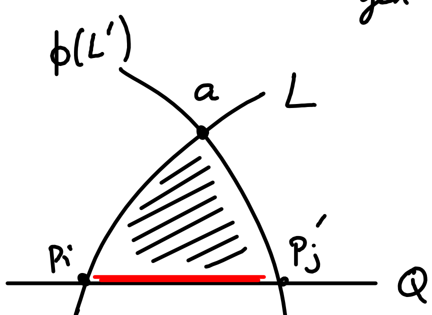
  $\partial \mathcal{M}(p_i,p_j) \text{ "=" } \coprod_k \mathcal{M}(p_i,p_k) \times \mathcal{M}(p_k,p_j)$

  (usual traj-breaking argument)

- given L and L', $CW^*(L,L') \longrightarrow Hom_*((P_L, D_L), (P'_L, D'_L))$, with $P_L = \oplus p_i$, $P'_L = \oplus p'_j$

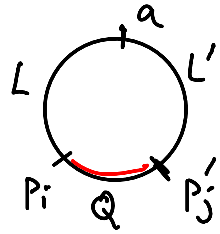
  $CW^*(L,L')$ gen'd by $a \in L \cap \phi(L')$

  $Hom_*((P_L, D_L), (P'_L, D'_L)) = \left(\bigoplus_{i,j} Hom_*(p_i,p'_j),\ \partial + D_L \circ - + - \circ D_{L'}\right)$



given by counting holom. disks

$$\mathcal{M}(p_i, p'_j, a) = \left\{ u : \right.$$



$$\left. \to M \,, \quad \bar\partial u = 0 \right\}$$

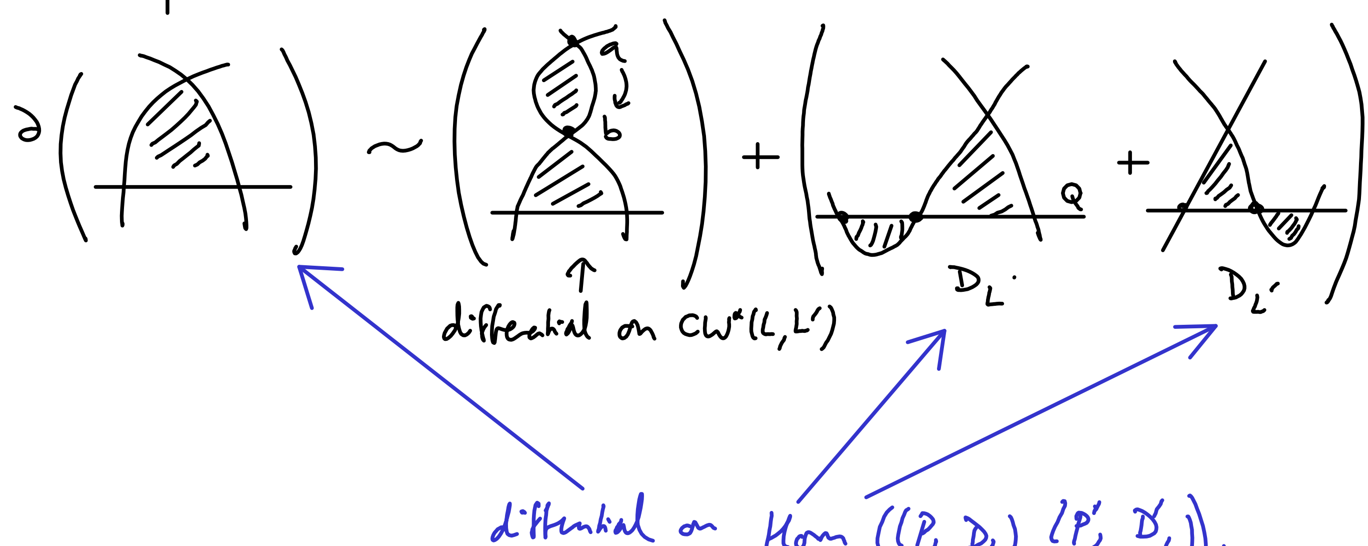
$$\mathcal{M}(p_i, p'_j, a) \xrightarrow{ev_\partial} \Omega_{p_i, p'_j} Q$$

Then our map $CW_*(L, L') \longrightarrow Hom_*((P_L, D_L), (P'_L, D'_L))$

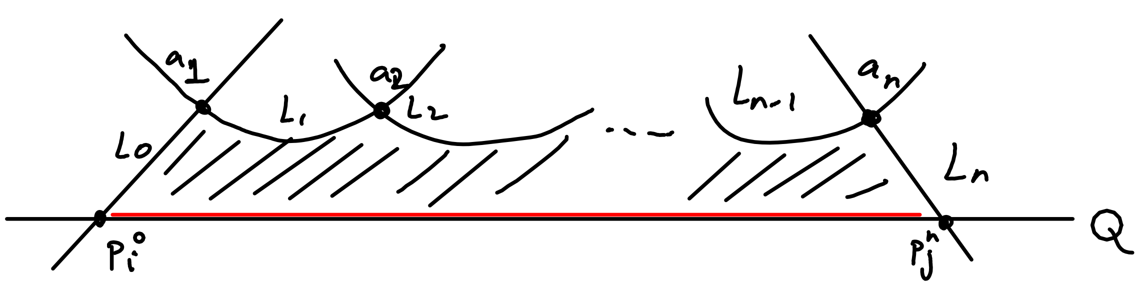
is given by $\quad a \longmapsto \{ ev_* [\mathcal{M}(p_i, p'_j, a)] \}_{i,j}$

<u>Claim:</u> this is a chain map

<u>Pf:</u>



- In fact, want an A_∞-functor, ie also need higher order maps

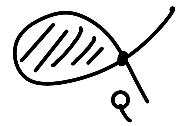


gives $CW^*(L_0, L_1) \otimes \cdots \otimes CW^*(L_{n-1}, L_n) \longrightarrow Hom_*((P_{L_0}, D_{L_0}), (P_{L_n}, D_{L_n}))$

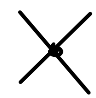
$\rightsquigarrow$ get A_∞-functor $\; \mathcal{W}(M) \to Tw\, \mathcal{P}(Q) \quad$ whenever $Q \subset M$ exact Lagr.

What if Q not embedded? (e.g. $Q = \bigcup Q_i \ldots$)

want: Q strongly exact Lagrangian immersion,

in particular $\nexists$  holomorphic.

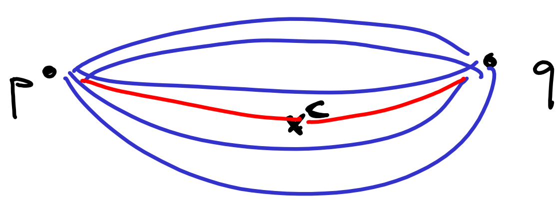
Cieliebak - Latschev ("relative SFT"): (Ng in dim. 1)

$\rightsquigarrow$ replace $C_*(\Omega_{p_i, p_j}(Q))$ to allow jumps at double points 

$$\mathrm{Hom}_*(p,q) = \Big(\bigoplus_{\substack{c_1 \ldots c_k \\ \text{self-intersections}}} C_*(\Omega_{p,c_1}) \otimes C_*(\Omega_{c_1,c_2}) \otimes \cdots \otimes C_*(\Omega_{c_k,q}),\ d = \partial + \delta \Big)$$

where δ involves string topology: generated by

$$\delta : C_*(\Omega_{p,q}) \longrightarrow \bigoplus_c C_*(\Omega_{p,c}) \otimes C_*(\Omega_{c,q})$$



select in given chain those loops that pass through a node c and break them there.

& then define an A_∞-functor as before ... ?
(allowing disks to have corners at double points c of Q)

* Problem: string topology operations not quite associative ....
$\Rightarrow$ chain level string topology = ???

* Some day, hope: can relax "strongly exact" assumption by incorporating corrections from discs purely bounded by Q.